On Anyonic Variables and Fractional Supersymmetry

E. Ahmed

Mathematics Department, Faculty of Sciences, Mansoura 35516, Egypt

Z. Naturforsch. **51 a,** 313–314 (1996); received January 31, 1996

A modified definition for anyonic variables is given. They are used to construct a representation for superspace of fractional supersymmetry in two dimensions. An extension to higher dimension is presented.

Introduction

The phenomena of fractional Hall effect [1, 2] and anyon superconductivity [3] have increased the interest in fractional symmetries. The attempts to apply this approach o supersymmetry [4] has been hampered by the lack of representations of variables which generalize Grassmann variables. Recently such variables have been defined [5, 6]. They are called anyonic variables.

In this work they will be slightly modified. It will be shown that they can be used as superspace variables for fractional supersymmetry (FSUSY) in $d = 2$. A nonrelativistic generalization to higher dimensions is discussed.

1. Anyonic Variables

The variables $\theta_1, \theta_2, \ldots$ are called anyonic variables of type q if

$$\theta_k \theta_j = q_{kj} \theta_j \theta_k , \tag{1}$$

$$q_{kj} = \exp[i q S(k-j)],$$

$$S(k-j) = \begin{cases} 1 & \text{if} \quad k > j , \\ -1 & \text{if} \quad k < j , \\ 0 & \text{if} \quad k = j . \end{cases} \tag{2}$$

Since $S(k-j) + S(j-k) = 0$, this definition is consistent. In the following only the case $q = 2\pi/n$, $n = 1, 2, 3, \ldots$ will be considered. In the previous work [5, 6] only the case $q = \pi/n$ has been studied. The case $n = 1$ corresponds to community variables. It will be shown that $n = 2$ anyonic variables are the known Grassmann variables.

The conjugate variables $\bar{\theta}_1, \bar{\theta}_2, \ldots$ satisfy the relation that, if $\eta\theta = \exp(iq)\,\theta\eta$, then $\eta\bar{\theta} = \exp(-iq)\,\bar{\theta}\eta$ and $\theta\bar{\theta} = \bar{\theta}\theta$.

Differentiation and integration are defined by analogy with Grassmann variables [7]:

$$\partial/\partial\theta_j(1) = 0,\ \partial/\partial\theta_j\,\theta_k f) = q_{jk}\,\theta_k\,\partial f/\partial\theta_j \quad \text{if} \quad k \neq j ,$$
$$\partial/\partial\theta_j(\theta_j f) = f + \exp(-iq)\,\theta_j\,\partial f/\partial\theta_j , \tag{3}$$

consequently

$$\partial(\theta_k)^p/\partial\theta_j = \delta_{kj}(\theta_k)^{p-1} \cdot (1 - \exp(2i\pi p/n))/(1 - \exp(2i\pi/n)) ,$$

which vanishes identically for $p = n$. Therefore the following condition is imposed on $q = 2\pi/n$, $n > 1$ anyonic variables:

$$(\theta_k)^n = 0 , \quad k = 1, 2, 3, \ldots . \tag{4}$$

This shows that $n = 2$ anyonic variables are identical to Grassmann ones and that anyonic variables generalize Grassmann ones to fractional ($n > 2$) cases. This generalization is not expected to be unique.

Using (4), anyonic integration is defined for $n > 1$ by

$$\int(\theta_j)^p\,\mathrm{d}\,\theta_k = 0\,\forall j,k \quad \text{if} \quad p < n-1 ,$$
$$\int(\theta_j)^{n-1}\,\mathrm{d}\,\theta_k = \delta_{jk} . \tag{5}$$

This definition guarantees translation invariance. For $n = 2$, Berezin formulas for Grassmann variables are regained.

2. Fractional Supersymmetry (FSUSY)

In this work it is claimed that $2\pi/n$ anyonic variables can be used to formulate a superspace-like construction for FSUSY. For $n = 2$ ordinary SUSY is regained. The case $d = 2$ will be discussed first. The generator Q for $d = 2$ FSUSY satisfies [4]

$$Q^n = P , \tag{6}$$

where P is the translation generator. The fractional superspace representation can be given, using a $2\pi/n$ anyonic variable θ, by

$$P = C_q\,\partial/\partial x , \quad C_q = \left[\prod_{j=0}^{n-2}(1 + q + q^2 + \ldots + q^j)\right],$$
$$Q = (C_q)^{(1/n)}\,D , \quad D = \partial/\partial\theta + (\theta)^{n-1}\,\partial/\partial x , \tag{7}$$

Reprint requests to Dr. E. Ahmed.

0932-0784 / 96 / 0400-0313 $ 06.00 © – Verlag der Zeitschrift für Naturforschung, D-72072 Tübingen

where $q = \exp(2\pi i/n)$. It is straightforeward to check (7) by taking into account both (4) and the relation

$$\sum_{k=1}^{n} q^k = 0 .$$

Having shown that anyonic variables can be used to construct a superspace-like formulation for $d = 2$ FSUSY, we will attempt a generalization to higher dimensions. In $(2 + 1)$ dimensions one can use two $2\pi/n$ anyonic variables θ_1, θ_2 and define

$$\begin{aligned} &P_1 = C_q \partial/\partial x , \quad P_2 = C_q \partial/\partial y , \\ &D_j = \partial/\partial\theta_j + (\theta_j)^{n-1} \partial/\partial x_j , \quad x_1 = x, x_2 = y , \\ &Q_j = (C_q)^{1/n} D_j . \end{aligned} \tag{8}$$

It can be shown that

$$(Q_j)^3 = P_j , j = 1, 2 \quad \text{and} \quad D_1 D_2 - q D_2 D_1 = 0 . \tag{9}$$

Generalization to $d > 3$ is straightforeward.

In order to get a relativistic formulation for (8) and (9), a generalization for the Dirac equation to the anyonic case should be found. This is a task for the future.

Acknowledgements

I wish to thank M. B. Sedra and the International Centre for Theoretical Physics where this work has been done.

[1] R. B. Laughlin, Phys. Rev. Lett. **60,** 2677 (1988).
[2] E. Fradkin, Field Theory of Condensed Matter Systems, Benjamin/Cummings 1991.
[3] F. Wilczek, Fractional Statistics & Anyon Superconductivity, World Scientific 1990.
[4] E. H. Saidi, M. B. Sedra, and J. Zerouaoui, Class. Quant. Grav. **12,** 1567 (1995).
[5] E. Ahmed, N. Maohammedi, and S. S. El-Mahdy, Phys. Lett A **183,** 277 (1993).
[6] E. Ahmed, A. S. Hegazi, and S. S. El-Mahdy, Int. J. Theor. Phys. **34,** (July ed.) (1995).
[7] J. G. Taylor and S. Ferrara, Supergravity 81, Cambridge Univ. Press 1982.